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The Minimally Nonplanar Graph of a Mexican Power Network

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Although the Mexican Power Network in 1990 contained just eleven influential politicians, the graph G, in which the edges indicate political alliances, was too complicated to show its structure clearly because it had 23 crossings of pairs of edges. In an attempt to simplify G, we found a drawing with only one crossing. We present a mathematical proof that G can not be simplified further.

Introduction

In order to analyze the structure of the Mexican Power Network, Gil-Mendieta, Schmidt, Castro and Ruiz (1997) displayed a sequence of graphs showing the evolution of the network kernel. The whole political network contains 5400 actors, and has an extremely complicated structure. The kernel contains only 39 powerful politicians **(Figure 1)** selected by following an historical criterion and the description of former presidents and other influential politicians. They showed the dynamics of the centrality in the kernel, i.e. its shifting over time since some actors disappeared (died) while others entered it. Using decade-long intervals, the changes in this network from 1920 to 1990 were shown.

The Mexican Power Network in 1990 contained just eleven influential politicians. They are represented by the nodes of the graph G of Figure 2, in which the edges indicate political alliances. The original drawing (part of Figure 1) was too complicated to show its structure clearly because it had 23 crossings

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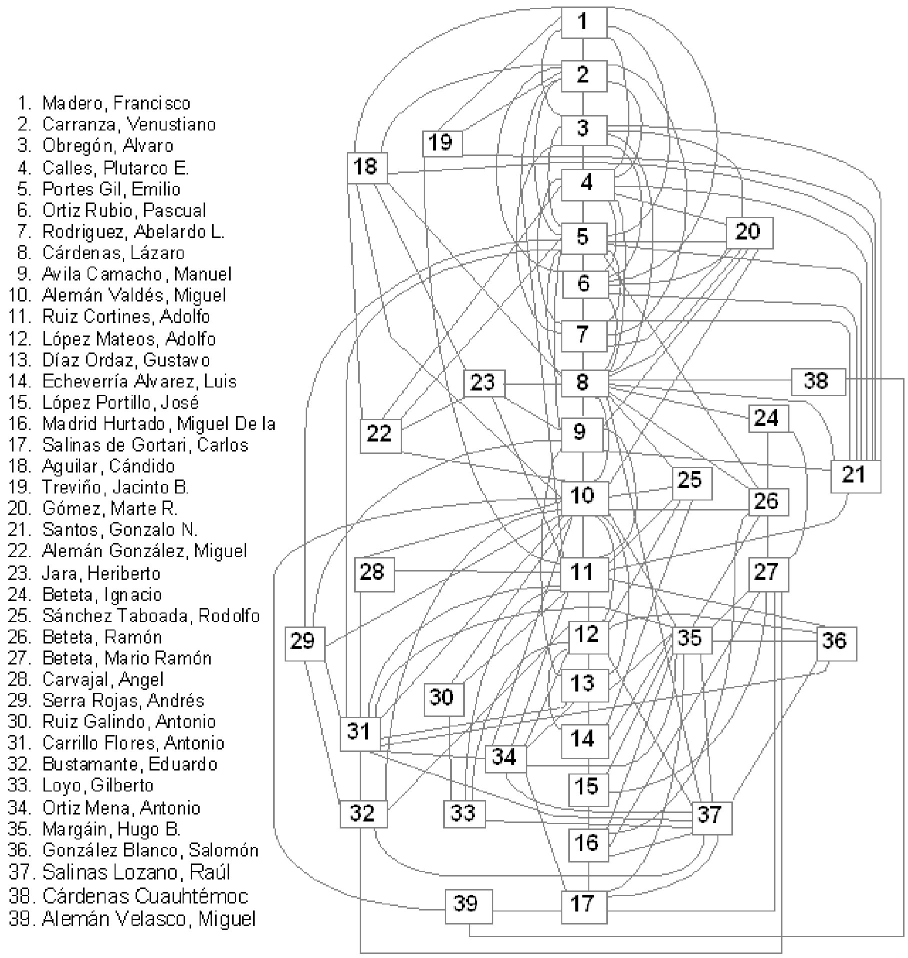

of pairs of edges. In an attempt to simplify ***G***, we finally found a drawing with only one crossing. We present a mathematical proof that ***G*** can not be simplified further. Graph theoretic concepts not presented here can be found in Harary (1969).

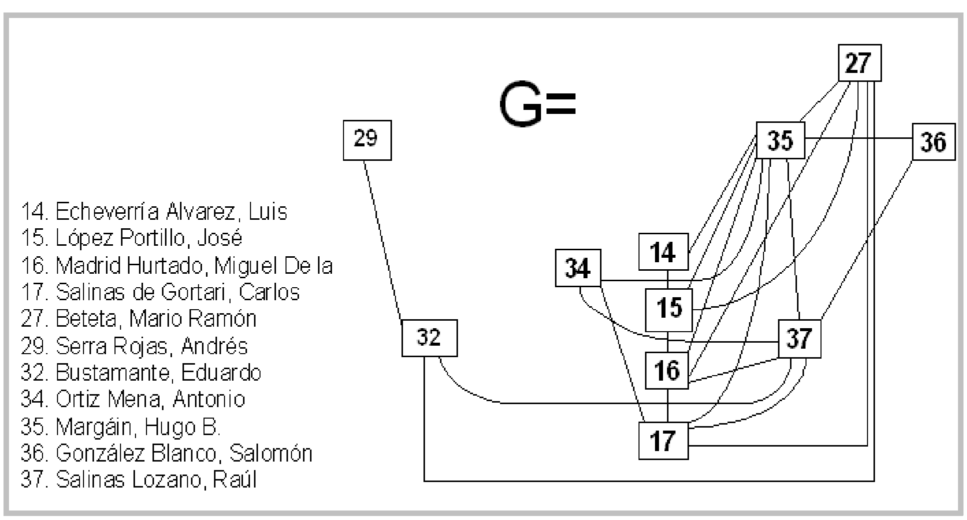

Figure 2. The Mexican Network of Power (1990) with 11 actors.

Redrawing the graph

Gil-Mendieta and Schmidt (1996), in an analysis of the creation and evolution of the Mexican Power Network (MPN), showed the chang es in the center of the network by analyzing and measuring the shift in centrality when new actors arrived and others left. Their Figure 1 shows the MPN graph with a very complex web of edges describing different connections among the 39 actors during 1922-1990.

The analysis was based on decade-long intervals, which included more than one presidential administration (six years). Thus it was possible to overcome limitations which could assign high values to the president due to his hierarchical position. Then they measured the centrality values of actors with extended influence (beyond the limits of one administration), using the formula (1) to calculate the value of the node index power or node centrality.

The *diameter* d(***G***) of a connected graph ***G*** is the maximum length of a geodesic (Harary 1969, p. 14).

The *node index power* of node ***v*** (Gil et al. 1997) is written A *subdivision of an edge* ***uv*** of ***G*** is obtained by inserting into this edge some new nodes of degree 2. A *subdivision of a graph* ***G*** (**Figure 6**) is the result of subdividing its edges. Further, ***G*** is also considered as a subdivision of itself.(v). Let n be the total number of nodes in the ***G*** and ***n***(***v***,***k***) the number of nodes at distance ***k*** from ***v***. Here ***1/k*** is the influence distance factor.

$$I(v) = \left[\sum_{k=1}^{d(G)} n(v,k)/k \right] \Big/ (n-1) \qquad (1)$$

The only node at distance zero from ***v*** is ***v*** itself. As ***G*** is connected, every node ***u*** ¹ v has a distance from ***v*** between 1 and ***d***(***G***), proving equation (2).

$$n = \sum_{k=0}^{d(G)} n(v,k) \qquad (2)$$

For simplicity we relabelled the nodes of ***G*** in Figure 2 and redrew it many times with fewer edge crossings each time, finally obtaining the graph ***H*** of Figure 3 with the minimum possible number, one, of crossings. This visual simplification enables both a clearer and more rapid understanding of the structure of this network and its centrality.

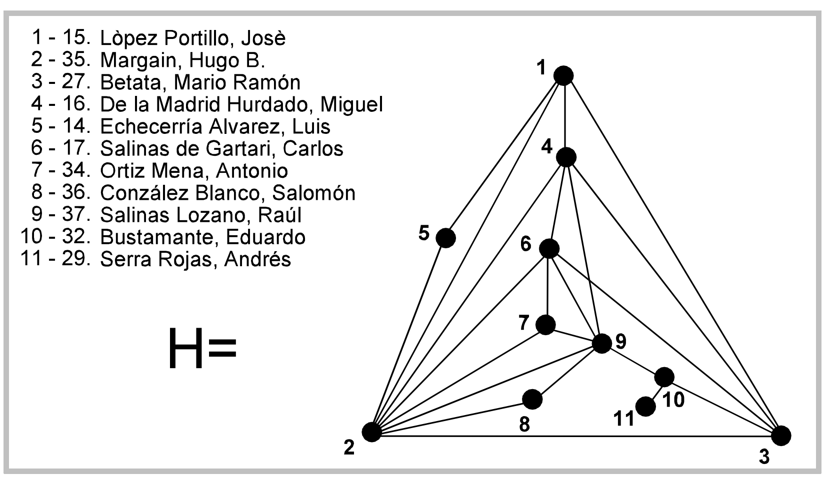

Figure 3. This relabeled graph with nodes marked 1 to 11, is isomorphic to Figure 2, drawn with just one crossing, namely edges 36 and 49. It is not planar and so is minimally nonplanar.

From Equation (1) we calculate the centrality of each node, in the political case we named it *node index power* and defines the extended influence power of each actor. For the graph of Figure 3, we get:

Table 1. $\boldsymbol{n(v,k)}$ and $\boldsymbol{I(v)}$ values.

	$n(v,k)$				$I(v)$
	k=4	k=3	k=2	k=1	
V_1	0	1	5	4	0.6833
V_2	0	1	1	8	0.8833
V_3	0	0	5	5	0.7500
V_4	0	1	4	5	0.7333
V_5	1	1	6	2	0.5583
V_6	0	1	4	5	0.7333
V_7	0	1	6	3	0.6333
V_8	0	1	7	2	0.5833
V_9	0	0	4	6	0.8000
V_{10}	0	1	6	3	0.6333
V_{11}	1	6	2	1	0.4250

Table 1 list both the degrees of the 11 nodes of Figure 3 under the column k=1 and the $I(v)$ values in the last column. The non-increasing degree sequence of G is (8, 6, 5, 5, 4, 3, 3, 2, 2, 1) for nodes 2, 9, 3. 4, 6, 1, 7, 10, 8, 5, 11 respectively. This is exactly the same sequence of nodes for the non-increasing sequence of values of the $I(v)$. Hence we see empirically that the degree sequence can be used as a close approximation to the power index the $I(v)$ sequence.

A *planar graph* (Harary 1969, p. 102) can be drawn in the plane without any crossings. A nonplanar graph must have at least one crossing.

A drawing of a planar graph in the plane without crossings results in a *plane graph*. The *crossing number* X(G) of a graph (Harary 1969, p.122) is the smallest possible number of crossings in a drawing of G in the plane. Thus X(G) = 0 if and only if G is planar. A graph G is called *minimally nonplanar* (Harary 1965) if its crossing number X(G) is 1.

We now develop the statement of the classical theorem that provides a criterion for a graph to be planar. Several definitions are required.

A node v and an edge e of a graph are *incident* when e joins v with some other node u, so that $e = uv$.

The *degree* of v is the number of edges incident with v, i.e. the number of nodes adjacent to v, see Figure 4.

The *removal of an edge* e from a graph F results in the subgraph $F - e$ that contains all the nodes of F and every edge except e (Harary 1969, p. 11).

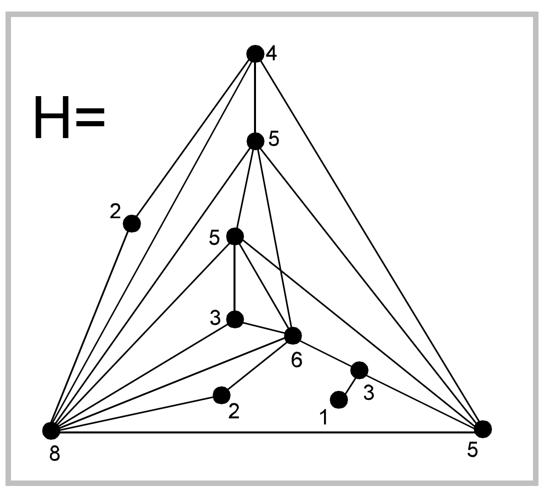

Figure 4. Degrees of the nodes of Figure 3.

To illustrate, Figure 5 shows that the two best known nonplanar graphs; the complete graph K_5 and the complete bipartite graph $K_{3,3}$ are minimally nonplanar.

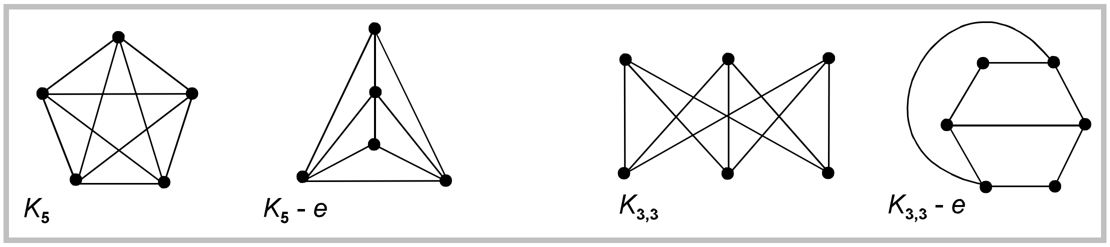

Figure 5. Two nonplanar graphs K_5 and $K_{3,3}$ and the planar subgraphs obtained by removing one edge from each.

A *subdivision of an edge* ***uv*** of ***G*** is obtained by inserting into this edge some new nodes of degree 2. A *subdivision of a graph* ***G*** (**Figure 6**) is the result of subdividing its edges. Further, ***G*** is also considered as a subdivision of itself.

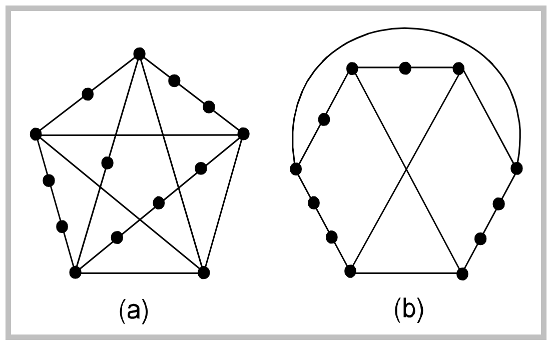

Figure 6. A subdivision (a) of K_5 and (b) of $K_{3,3}$

Kuratowski's Theorem (see Harary 1969, p. 108). A graph ***G*** is planar if and only if ***G*** contains no subgraph which is a subdivision of $\boldsymbol{K_5}$ or of $\boldsymbol{K_{3,3}}$.

With the help of this powerful criterion for the planarity of a graph, we are now ready to prove our result.

Theorem. The graph ***G*** of Figure 2 is minimally nonplanar.

Proof. Figure 7, which is a subdivision of $K_{3,3}$, is a subgraph of Figure 3.

Therefore Figure 3 is nonplanar by Kuratowski's Theorem. But graphs ***H*** of Figure 3 and ***G*** of Figure 2 are isomorphic and ***H*** has only one crossing. Thus ***G*** is minimally nonplanar.

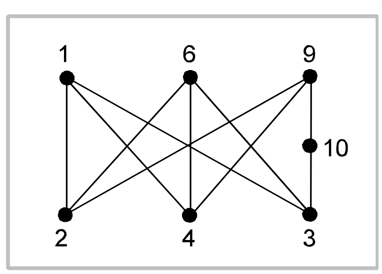

Figure 7. A subgraph of G in Figure 3. It is a subdivision of $K_{3,3}$.

The political analysis implications

Redrawing the graph loses the chronological order, which is important because political actors have historical presence. However, the relative chronological order can be represented by a **digraph**, where the arrows show which actor has precedence and bi-directional arrows show that the two actors are contemporary (**Figure 8**). This doesn't solve the issue of the actor's hierarchical position, which also can be shown by arrows of a different type.

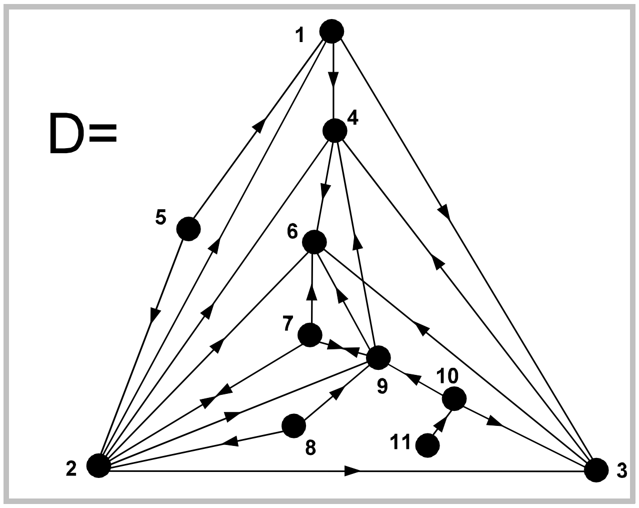

Figure 8. A digraph whose graph is G.

An actor with the maximum degree will have more influence than those with lesser degree since he has as many direct contacts as possible. In the given network, the actors with maximum degree are #2 with degree 8 and #9 with degree 6 **(Figure 4).** A higher hierarchical position means formal disposition of political resources (Dahl 1963) which combined with seniority can produce the relative political relevance of a political actor.

The *degree sequence* of a graph is a list of all the degrees from the largest to the smallest. For ***H*** in Figure 4 , this sequence is (8,6,5,5,5,4,3,3,2,2,1). The unique actor in Figure 3 with most direct connections is #2.

Conclusion

Redrawing the nonplanar graph to display it with fewer crossings enables clearer understanding of its structure. The simplification has visual intuitive and explanatory advantages, even though it produced a (minimally) nonplanar graph.

However, from the political analysis perspective, redrawing the graph could lose the chronological - perspective and also the power relation because of the change in time. In particular in Figure 8, nodes 1,4,5 and 6 were presidents. Therefore the direction of influence depends on the year of the analysis.

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